Zaslavsky's Theorem on acyclic orientations of signed graphs

Definitions:
Σ is a signed graph
σ is the function that assigns +1 or -1 to each edge
k is a signed coloring of Σ, k is proper if $k(v) - \sigma(e)k(w)$ is nonzero for all $e = vw$
τ is an orientation of Σ, each half edge is assigned either +1 or -1. +1 means the arrow points toward the vertex, -1 means the arrow points away from the vertex. τ satisfies $\tau(v,e)*\tau(w,e) = -\sigma(e)$ for each $e = vw$
χ is the signed chromatic polynomial, $\chi(\Sigma,2\lambda+1)$ counts how many proper colorings in λ signed colors there are on Σ.
χ^b is the zero free (or balanced) signed chromatic polynomial, χ^b(Σ,2λ) counts how many proper colorings in λ signed colors with none of these colors being zero, there are on Σ

$\tilde{\Sigma}$ is the covering graph of Σ. For each vertex v in Σ, there are vertices +v and -v in $\tilde{\Sigma}$. For each edge in Σ, there are edges $\tilde{e}_1$ and $\tilde{e}_2$ in $\tilde{\Sigma}$.
p is the projection from $\tilde{\Sigma}$ to Σ. $p(\varepsilon v) = v$ where $\varepsilon = \pm$, $p(\tilde{e}_1) = p(\tilde{e}_2) = e$
p^-1 is the lifting function from Σ to $\tilde{\Sigma}$. p^-1(v) = {+v,-v}, p^-1(e) = {$\tilde{e}_1$,$\tilde{e}_2$}
The covering graph satisfies the following: e = vw lifts to $\tilde{e}_1 = (\varepsilon_1 v)(\varepsilon_2 w)$ and $\tilde{e}_2 = (-\varepsilon_1 v)(-\varepsilon_2 w)$
With $\varepsilon_1\varepsilon_2 = \sigma(e)$
$\tilde{k}$ is the lifted coloring on $\tilde{\Sigma}$, $\tilde{\tau}$ is the lifted orientation which satisfy the following
$\tilde{k}(\varepsilon v) = \varepsilon k(v)$ and $\tilde{\tau}(\varepsilon v,\tilde{e}) = \varepsilon\tau(v,e)$
It is worth noting that on Σ all edges are positive and the orientation $\tilde{\tau}$ has both half edges pointing in the same direction

A cycle in a signed graph is a set of vertices and edges such that they form a closed loop in the underlying unsigned graph and for each vertex in the cycle, there is an arrow pointing toward the vertex and an arrow pointing away from out.
C(Σ) is the cyclic part of Σ, the union of all cycles in Σ

I(k) is the set of all improper edges in Σ under k, i.e. the set of all edges e = vw in Σ, such that $k(v) = \sigma(e)k(w)$

Switching Σ by υ means that υ assigns +1 or -1 to each vertex of Σ and k,τ and σ are replaced by k^υ,τ^υ and σ^υ where if e = vw then $k^\upsilon(v) = \upsilon(v)k(v)$, $k^\upsilon(w)=\upsilon(w)k(w)$,
$\sigma^\upsilon(e) = \upsilon(v)\upsilon(w)\sigma(e)$ and $\tau^\upsilon(v,e) = \upsilon(v)\tau(\omega,\varepsilon)$

A cycle is positive if the product of the signs of its edges is positive. An edge set is balanced if every cycle in it is positive

Let S be a subset of the edges of Σ. Then Σ/S is the graph constructed by first switching Σ and k, so that every balanced component of S has all positive edges, then performing unsigned contraction on the edges of S. Σ/S is defined up to switching.

k’ is the induced coloring and τ’ the induced orientation on Σ/S. k’ = k on vertices unaffected by contraction and k’ assigns one of the colors of the original vertices to the new vertex that is created by contraction.

The following statements we will assume without proof. The proofs are all straightforward, but some are lengthy and tedious.

Lemma 1: $C(\tilde{\tau}) = p^{-1}(C(\tau))$ which is to say that when a cycle in Σ is lifted, it becomes one or more cycles in $\tilde{\Sigma}$, and when a cycle in $\tilde{\Sigma}$ is projected, it becomes part of a cycle in Σ

This is proved by considering a positive cycle $v_1 \to v_2 \to \dots \to v_n \to v_1$ in Σ and noticing that it lifts to two distinct cycles in $\tilde{\Sigma}$, a non positive cycle in Σ lifts to a single cycle in $\tilde{\Sigma}$ and that a cycle in $\tilde{\Sigma}$ projects to a single cycle in Σ

Lemma 2: $I(\tilde{k}) = p^{-1}(I(k))$ which is to say that an improper/proper edge remains improper/proper when lifted or projected. This follows directly from the definitions

Lemma 3: $I(k^{\upsilon}) = I(k)$ which is to say that switching an edge does not change whether or not it is improper. This follows from the definitions by considering when one, both or neither of two adjacent vertices are assigned -1 by υ

Lemma 4: I(k) is balanced if k is zero free and every vertex, v, in an unbalanced component of I(k) must have k(v)=0
This can be proven by assuming that I(k) has a nonpositive cycle, $v_1 \to v_2 \to \dots \to v_n \to v_1$ and using the fact that all of the edges connecting these vertices are improper to show that $k(v_1) = -k(v_1)$ meaning that $k(v_1)=0$. All vertices in the same component of I(k) must have the same color up to sign.

Lemma 5: There is a bijection between colorings k of Σ and proper colorings, k’ of contractions of Σ. I.e. For every coloring k, a unique induced coloring k’ can be constructed by switching and contracting Σ into Σ/I(k). Similarly, given a proper coloring k’ of Σ/A, a unique coloring k of Σ can be constructed such that A = I(k). In addition, the induced coloring k’ of I(k) is proper. Also, if k is zero free, then the induced coloring k’ of Σ/I(k) is zero free and vice versa.
k’ can be constructed from k by switching k and Σ until I(k) is positive, then contracting Σ. k can be constructed from k’ by k=k’ on balanced components of I(k), coloring unbalanced components 0, and then reversing the switching on Σ.

The Proof begins here

Just like in the proof of Stanley's Theorem, we begin by constructing a unique orientation τ, for every proper coloring k, of Σ, using the rule

$$\tau(v,e)k(v)+\tau(w,e)k(w)>0$$ Call this property $(*)$

Using the fact that

$$\sigma(e)=-\tau(v,e)\cdot\tau(w,e)$$

This becomes

$$\tau(v,e)\cdot\big(k(v)-\sigma(e)k(w)\big)>0$$

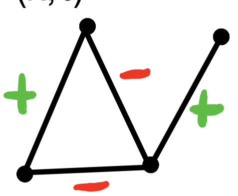
In other words, define τ by

$$\tau(v,e)=\frac{(k(v)-\sigma(e)k(w))}{|k(v)-\sigma(e)k(w)|}=\mathrm{sign}\big(k(v)-\sigma(e)k(w)\big)$$

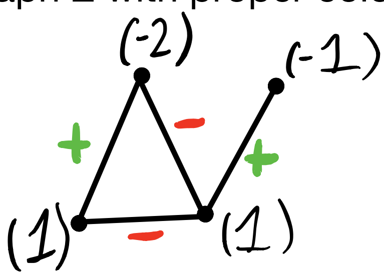
If k and τ satisfy $(*)$ for Σ, then call (k,τ) a proper pair
If k is proper then k(v)-σ(e)k(w) is non zero so τ is well defined, (k,τ) is a proper pair and τ is unique by construction.
If (k,τ) is a proper pair, then τ(v,e)*(k(v)-σ(e)k(w))>0 so k(v)-σ(e)k(w) is nonzero so k is proper on e. Therefore for every proper coloring, k, of Σ, there is one and only one proper pair (k,τ)
So there are the same number of proper pairs as there are proper colorings, therefore χ(2λ+1) = the number of proper pairs where k is a coloring in λ colors, and (χ^b)(2λ) = the number of proper pairs where k is a zero-free coloring in λ colors
It is worth noting that by lifting the coloring k on Σ and the orientation τ on Σ to $\tilde{k}$ and $\tilde{\tau}$ on $\tilde{\Sigma}$, property $(*)$ is preserved and it is identical to Stanley's definition of proper pair when each edge has both of its arrows pointing the same way. Therefore (k,τ) is a proper pair iff $(\tilde{k},\tilde{\tau})$ is.

For example:

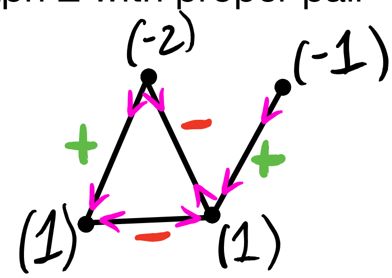
Graph Σ



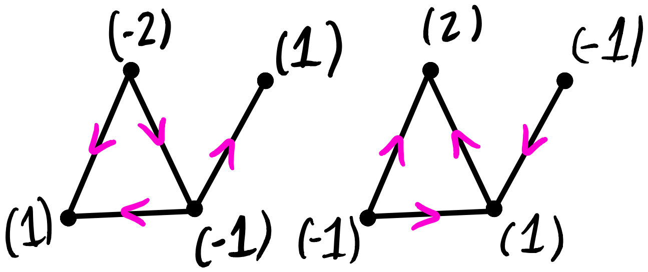
Graph Σ with proper coloring k

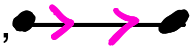


Graph Σ with proper pair (k,τ)



Covering graph Σ with proper pair (k,τ)

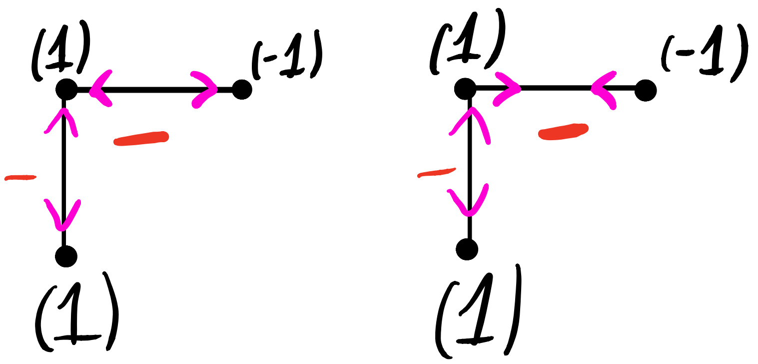


For convenience, when discussing the covering graph,  is written as 

In analogy with Stanley's proof we define a compatible pair, (k,τ) as a pair that satisfies

$$\tau(v,e)k(v)+\tau(w,e)k(w) \geq 0$$ Call this property $(*)'$

The definition of compatible pairs is the same as the definition of proper pairs except that it allows for improper colorings. It is important to note that if k is improper on edge e, then it has two compatible orientations that differ only on that edge.
For example:



The same coloring is compatible with different orientations

Define $\overline{\chi}_{\Sigma}(2\lambda+1)$ to be the number of compatible pairs (k,τ) of Σ, where τ is acyclic

It's very straightforward to show that $(\tilde{k},\tilde{\tau})$ is compatible iff (k,τ) is

To proceed further, we must first show that switching vertices, and the colors on those vertices, by υ does not affect compatibility

Note that if e = vw then

$\upsilon(v)=+1, \upsilon(w)+1 \Rightarrow$ Nothing changes

$\upsilon(v)=-1, \upsilon(w)=+1 \Rightarrow k^{\upsilon}(v)=-k(v), k^{\upsilon}(w)=k(w)$

$\sigma^{\upsilon}(e)=-\sigma(e)$ and $\tau^{\upsilon}(v,e)=-\tau(v,e)$

So $\tau^{\upsilon}(v,e)\cdot(k^{\upsilon}(v)-\sigma^{\upsilon}(e)k^{\upsilon}(w))=-\tau(v,e)(-k(v)+\sigma(e)k(w))$

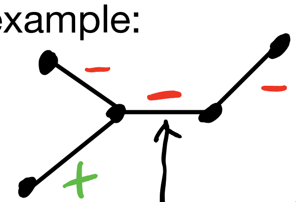
$=\tau(v,e)\cdot(k(v)-\sigma(e)k(w))$ compatibility is unchanged

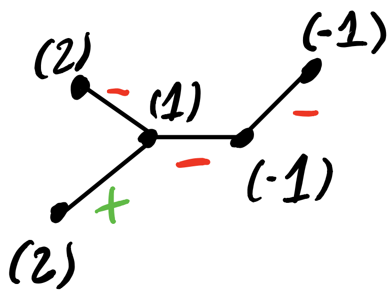
$\upsilon(v)=+1, \upsilon(w)=-1 \Rightarrow$ Same as case above

$\upsilon(v)=-1, \upsilon(w)=-1 \Rightarrow \kappa^{\upsilon}(v)=-\kappa(v), \kappa^{\upsilon}(w)=-\kappa(w),$
$\sigma^{\upsilon}(e)=\sigma(e)$ and $\tau^{\upsilon}(v,e)=-\tau(v,e)$
So $\tau^{\upsilon}(v,e)\cdot(\kappa^{\upsilon}(v)-\sigma^{\upsilon}(e)\kappa^{\upsilon}(w))=-\tau(v,e)(-\kappa(v)+\sigma(e)\kappa(w))$
$=\tau(v,e)\cdot(\kappa(v)-\sigma(e)\kappa(w))$ compatibility is unchanged

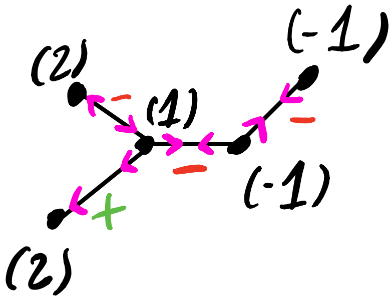
Therefore switching does not affect compatibility.

It is also necessary to note that contraction does not affect compatibility so long as the contracted edge is improper. This is clear since, after switching, both vertices of the contracted edge have the same color.
For example:

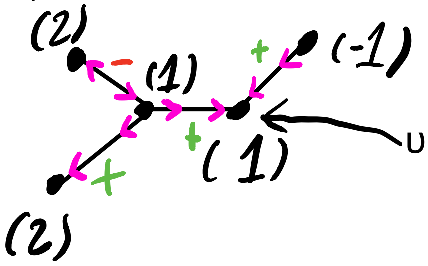
$\Sigma =$ 
call this edge e

, Σ with improper coloring κ 

Σ with compatible pair (κ, τ)

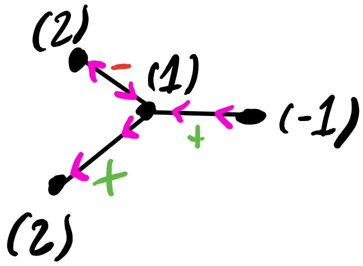


Σ with compatible pair (κ, τ) after switching by υ so that e is positive



υ gives this vertex -1 and the rest +1

Σ/e with induced coloring k' and orientation τ'



Lemma 6:

Let S be a subset of I(k), let k' and τ' be the induced coloring and orientation of Σ/S, then (k,τ) is compatible iff (k',τ') is. In addition, if S = I(k) then (k,τ) is compatible iff (k',τ') is proper.
The first part is easy to see since k' is constructed by switching and contracting and both switching and contracting preserve compatibility. The second part follows from the fact that if k' is the induced coloring of Σ/I(k), then k' is proper.

Lemma 7:

Let (k,τ) be a compatible pair for Σ. Then C(τ) is a subset of I(k), which is to say that every edge that is a part of a cycle is improper. In addition, if k is proper, then τ is acyclic
This can be proved by lifting Σ to $\tilde{\Sigma}$ then applying a lemma from the proof of Stanley's theorem.
Assume that $\tilde{\tau}$ contains a cycle, call it $\tilde{V}_1 \to \tilde{V}_2 \to \dots \to \tilde{V}_n \to \tilde{V}_1$
Then by the condition for compatible pairs, we must have that

$$k(\tilde{V}_1) \leq k(\tilde{V}_2) \leq \dots \leq k(\tilde{V}_n) \leq k(\tilde{V}_1)$$

Thus $k(\tilde{V}_1) = k(\tilde{V}_2) = \dots = k(\tilde{V}_n) = k(\tilde{V}_1)$

Every vertex in this cycle is colored the same color, therefore every edge in this cycle is improper. In other words,

$$C(\tilde{\tau}) \subseteq I(\tilde{k})$$

By lemmas 1 & 2, we have

$$C(\tilde{\tau}) = \rho^{-1}(C(\tau)) \text{ and } I(\tilde{k}) = \rho^{-1}(I(k))$$

$$\text{So } \rho^{-1}(C(\tau)) \subseteq \rho^{-1}(I(\tilde{k})), \text{ so } C(\tau) \subseteq I(k)$$

If k is proper, then I(k) is empty so C(τ) is empty, meaning that τ is acyclic.

We now wish to create a bijection between compatible pairs of Σ and compatible pairs of Σ/e and $\Sigma\backslash e$ for every e in Σ

Let e be an edge of Σ, k a coloring of Σ and k' the induced coloring of Σ/e if the induced coloring exists. Note that k' exists iff e is improperly colored by k. Let τ' be the induced orientation on Σ/e

Define τ_e to be τ except with the arrows of edge e reversed. Define $\tau\backslash e$ to be an orientation on $\Sigma\backslash e$ that is equal to τ on edges other than e

There are 3 cases:

(1) Both (k,τ) and (k,τ_e) are compatible. Then e must be improperly colored by k since

$$\tau(v,e)\cdot(k(v)-\sigma(e)k(w))\geq 0 \text{ and } \tau_e(v,e)\cdot(k(v)-\sigma(e)k(w))$$
$$= -\tau(v,e)\cdot(k(v)-\sigma(e)k(w))\geq 0, \text{ so } k(v)=\sigma(e)k(w)$$

e must be the only improper edge so $(k,\tau\backslash e)$ is compatible. e is improper, so by lemma 6, (k',τ') is compatible.

(2) One of (k,τ) and (k,τ_e) is compatible. Then k must be proper on e, so k' does not exist. k is proper on all edges of Σ, so $(k,\tau\backslash e)$ is compatible.

(3)Neither (k,τ) nor (k,τ_e) is compatible. In this case there must be some edge other than e that is improper under k. Therefore neither of $(k,\tau/e)$ and (k',τ') are compatible.

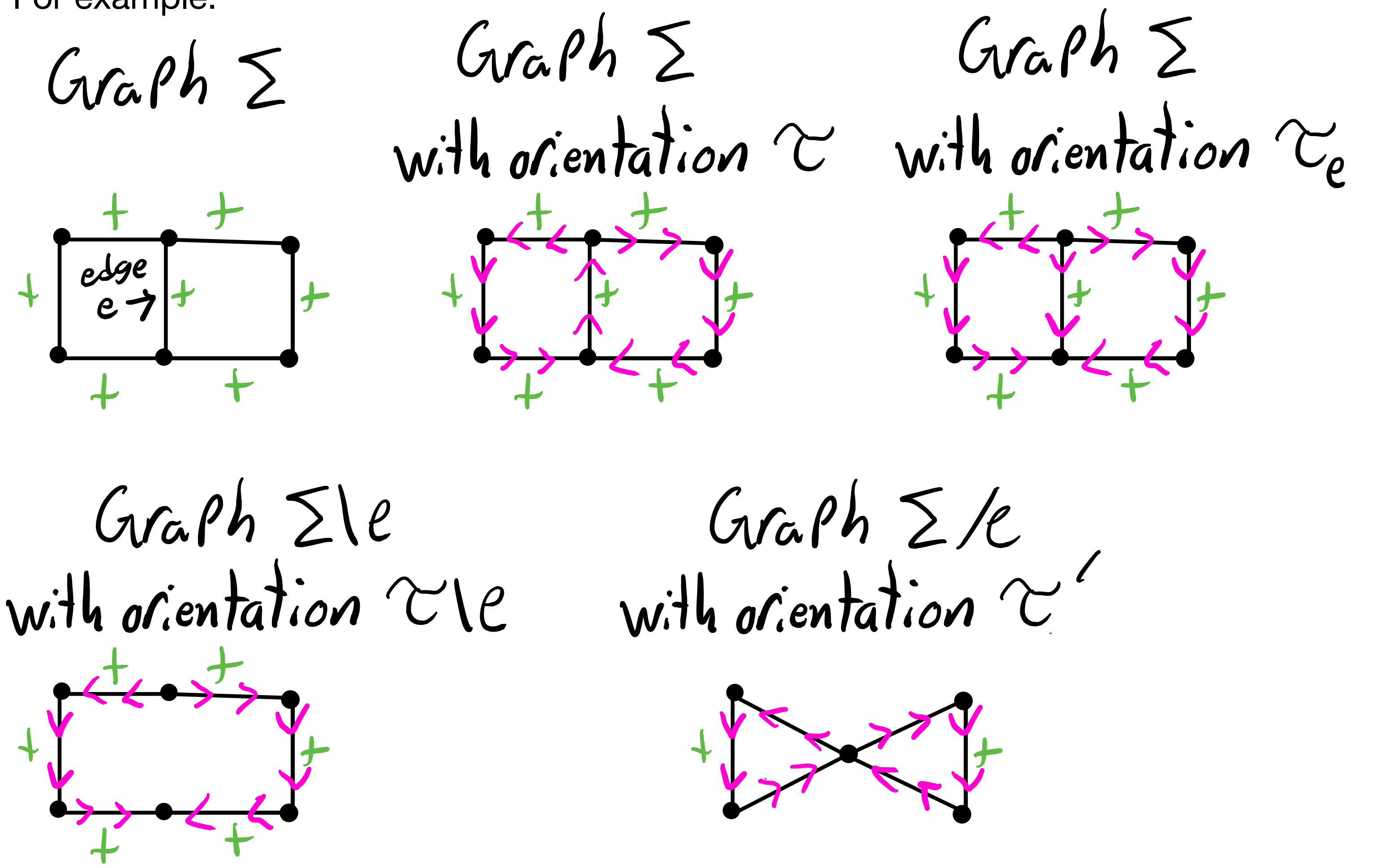
In short, we have the following table:

compatible in Σ	compatible in $\Sigma\backslash e$ or Σ/e
Both (k,τ) and (k,τ_e) =>	Both $(k,\tau\backslash e)$ and (k',τ')
one and only one of (k,τ) and (k,τ_e) =>	only $(k,\tau\backslash e)$
Neither =>	Neither

Similarly to the proof of Stanley's theorem, we wish to count compatible pairs where the orientation is acyclic.

In case 1, if one of the pairs is acyclic and the other is not (Wlog τ contains a cycle), it must be true that e is an element of every cycle in τ. Therefore removing e breaks all of these cycles while contracting e keeps these cycles intact.

For example:



In this example τ_e is acyclic but τ is not. This forces τ\e to be acyclic and τ' to be cyclic.

If both τ and τ_e are acyclic, then deleting or contracting and edge cannot introduce a cycle, so both τ\e and τ' are acyclic.

If both τ and τ_e are cyclic, then there is a cycle that does not involve e, so both τ\e and τ' are also cyclic.

In case 2, (Wlog τ is compatible and τ_e is not) if τ is acyclic then deleting e cannot create a cycle, so τ\e is acyclic. If τ\e is acyclic then at least one of τ and τ_e must be acyclic. Remember that e must be proper in case 2. Assume τ is cyclic, by lemma 7, e must be unproper. Contradiction. Therefore τ is acyclic iff τ\e is.

Now, by looking at the cases, we make the observation that for each acyclic compatible pair of Σ there is one and only one acyclic compatible pair of Σ\e or Σ/e

Let $\overline{\chi}_{\Sigma}$ be the number of acyclic compatible pairs of Σ. We have just shown that

$$\overline{\chi}_{\Sigma}(2\lambda+1) = \overline{\chi}_{\Sigma\setminus e}(2\lambda+1) + \overline{\chi}_{\Sigma/e}(2\lambda+1)$$

It is clear from the definitions that $\overline{\chi}_{\bullet}(2\lambda+1) = 2\lambda+1$ and that $\overline{\chi}_G * \overline{\chi}_H = \overline{\chi}_{G \sqcup H}$

It can be shown by the same methods as in the unsigned case that
$\chi_{\Sigma}(2\lambda+1) = \chi_{\Sigma\setminus e}(2\lambda+1) - \chi_{\Sigma/e}(2\lambda+1)$

Also $\chi_{\bullet}(2\lambda+1) = 2\lambda+1$ and $\chi_G * \chi_H = \chi_{G \sqcup H}$ where χ is the signed chromatic polynomial.

Now we can show that $\overline{\chi}_{\Sigma}(2\lambda+1) = (-1)^n * \chi_{\Sigma}(-(2\lambda+1))$ by the exact same argument that was used to prove Stanley's theorem, where n is the number of vertices.

Similarly by only considering zero free colorings, and using the fact that if k is zero free, then the induced coloring k' on Σ/I(k) is also zero free, we can define $\overline{\chi}^{b}_{\Sigma}$ to be the number of acyclic compatible pairs of Σ with k zero free and we can repeat all of these steps to find that $\overline{\chi}^{b}_{\Sigma}(2\lambda) = (-1)^n * \chi_{\Sigma}(-2\lambda)$

Note that when λ=0 in the non-zero free case, we have that $\chi_{\Sigma}(0)$ is the number of acyclic orientations, since there is only one way to color everything with 0.

Therefore the number of acyclic orientations of a signed graph Σ is $(-1)^n * \chi_{\Sigma}(-1)$

For anyone who wants to see it, here is the full inductive argument, where p is the number of vertices and E the number of edges of a graph G

Let λ be odd. $\overline{\chi}_G(\lambda) = (-1)^p \chi_G(-\lambda)$

Let G be a single vertex with no edges

Then $\overline{\chi}_G(\lambda) = \lambda$ and $\chi_G(-\lambda) = -\lambda$

so $\overline{\chi}_G(\lambda) = (-1)^1 \chi_G(-\lambda)$

Assume the formula holds for all graphs with $p + E \leq n$. Let G be a graph with $p + E = n+1$ with $p > 1$ and $E \neq 0$

Then $\overline{\chi}_G(\lambda) = \overline{\chi}_{G-e}(\lambda) + \overline{\chi}_{G/e}(\lambda)$

and $\chi_G(-\lambda) = \chi_{G-e}(-\lambda) - \chi_{G/e}(-\lambda)$

$G-e$ has p verticies and $E-1$ edges, so $p + (E-1) = n$, so $\overline{\chi}_{G-e}(\lambda) = (-1)^p \chi_{G-e}(-\lambda)$ by the induction hypothesis.

G/e has $P-1$ verticies and $E-1$ edges so

$(P-1)+(E-1) \leq n$, so $\overline{\chi}_{G/e}(\lambda) = (-1)^{P-1}\chi_{G/e}(-\lambda)$

Therefore $\left(\overline{\chi}_{G-e}(\lambda)\right) + \left(\overline{\chi}_{G/e}(\lambda)\right)$

$= \left((-1)^{P}\chi_{G-e}(-\lambda)\right) + \left((-1)^{P-1}\chi_{G/e}(-\lambda)\right)$

$= (-1)^{P}\left(\chi_{G-e}(-\lambda) - \chi_{G/e}(-\lambda)\right)$

so $\overline{\chi}_{G}(\lambda) = \overline{\chi}_{G-e}(-\lambda) - \overline{\chi}_{G/e}(-\lambda)$

$= (-1)^{P}\left(\chi_{G-e}(-\lambda) - \chi_{G/e}(-\lambda)\right)$

$= (-1)^{P}\chi_{G}(-\lambda)$